2026 **年慶應義塾大学経済学部問題** *2*

数列 $\{a_n\}, \{b_n\}$ を次のように定めます。 $a_1 = 4\,,\ b_1 = 3$
$n = 2m-1$ $(m = 1, 2, 3, \cdots)$ のとき $a_{n+1} = -\dfrac{6}{5}a_n - \dfrac{9}{10}b_n\,, \quad b_{n+1} = \dfrac{9}{10}a_n - \dfrac{6}{5}b_n$
$n = 2m$ $(m = 1, 2, 3, \cdots)$ のとき $a_{n+1} = -\dfrac{3}{2}a_n\,, \quad b_{n+1} = \dfrac{3}{2}b_n$

$c_n = \left(\dfrac{4}{3}\right)^{n-1} a_n b_n$ とします。$S_n = \displaystyle\sum_{i=1}^{n} c_i$ を求めてください。

解説・解答

数列を何項か計算します。
$\{a_n\} = \{4, -\frac{15}{2}, \frac{45}{4}, -\frac{27}{2}, \cdots\}\,,\ \{b_n\} = \{3, 0, 0, \frac{81}{8}, \cdots\}\,,\ \{c_n\} = \{12, 0, 0, -4 \cdot 81, \cdots\}$

$a_{2m} = -\frac{6}{5}a_{2m-1} - \frac{9}{10}b_{2m-1}$
$b_{2m} = \frac{9}{10}a_{2m-1} - \frac{6}{5}b_{2m-1}$

$a_{2m+1} = -\frac{3}{2}a_{2m} = \frac{9}{5}a_{2m-1} + \frac{27}{20}b_{2m-1}$
$b_{2m+1} = \frac{3}{2}b_{2m} = \frac{27}{20}a_{2m-1} - \frac{9}{5}b_{2m-1}$

$a_{2m+2} = -\frac{6}{5}a_{2m+1} - \frac{9}{10}b_{2m+1} = -\frac{27}{8}a_{2m-1}$
$b_{2m+2} = \frac{9}{10}a_{2m+1} - \frac{6}{5}b_{2m+1} = \frac{27}{8}b_{2m-1}$

$a_{2m+3} = -\frac{3}{2}a_{2m+2} = \frac{81}{16}a_{2m-1}$
$b_{2m+3} = \frac{3}{2}b_{2m+2} = \frac{81}{16}b_{2m-1}$

$a_{2m+4} = -\frac{6}{5}a_{2m+3} - \frac{9}{10}b_{2m+3} = \frac{81}{16}\left(-\frac{6}{5}a_{2m-1} - \frac{9}{10}b_{2m-1}\right) = \frac{81}{16}a_{2m}$
$b_{2m+4} = \frac{9}{10}a_{2m+3} - \frac{6}{5}b_{2m+3} = \frac{81}{16}\left(\frac{9}{10}a_{2m-1} - \frac{6}{5}b_{2m-1}\right) = \frac{81}{16}b_{2m}$

よって n が奇数でも偶数でも $a_{n+4} = \frac{81}{16}a_n\,,\ b_{n+4} = \frac{81}{16}b_n$ です。
$c_{n+4} = (\frac{4}{3})^{(n+4)-1}a_{n+4}b_{n+4} = (\frac{4}{3})^{n+3}(\frac{81}{16}a_n)(\frac{81}{16}b_n) = 81 \cdot (\frac{4}{3})^{n-1}a_nb_n = 81c_n$
k を自然数として $(c_{4k-3}, c_{4k-2}, c_{4k-1}, c_{4k}) = (12 \cdot 81^{k-1}, 0, 0, -4 \cdot 81^k)$ です。

以上より
$n = 4k$ のとき $S_n = S_{4k} = \sum_{i=1}^{k}(c_{4i-3} + c_{4i-2} + c_{4i-1} + c_{4i}) = \sum_{i=1}^{k}(12 \cdot 81^{i-1} - 4 \cdot 81^i) = \frac{39(1-81^k)}{10}$
$n = 4k-1, 4k-2, 4k-3$ のとき $S_n = S_{4k} - c_{4k} = \frac{39(1-81^k)}{10} - (-4 \cdot 81^k) = \frac{39+81^k}{10}$