

2020 年横浜市立大学医学部問題 4

定積分 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 \theta}{\sin^3 \theta} d\theta$ の値を求めてください。

解説・解答

$$\begin{aligned}
 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin \theta} d\theta &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin \theta}{\sin^2 \theta} d\theta \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin \theta}{1 - \cos^2 \theta} d\theta \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin \theta}{(1 - \cos \theta)(1 + \cos \theta)} d\theta \\
 &= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{\sin \theta}{1 - \cos \theta} + \frac{\sin \theta}{1 + \cos \theta} \right) d\theta \\
 &= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{(1 - \cos \theta)'}{1 - \cos \theta} + \frac{(1 + \cos \theta)'}{1 + \cos \theta} \right) d\theta \\
 &= \frac{1}{2} \left[\log |1 - \cos \theta| - \log |1 + \cos \theta| \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \frac{1}{2} \left[\log \frac{1 - \cos \theta}{1 + \cos \theta} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \frac{1}{2} \left(\log \frac{1 - 0}{1 + 0} - \log \frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}} \right) \\
 &= -\log (\sqrt{2} - 1)
 \end{aligned}$$

$$\begin{aligned}
 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 \theta}{\sin^3 \theta} d\theta &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos \theta}{\sin^3 \theta} \cos \theta d\theta \\
 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos \theta}{\sin^3 \theta} (\sin \theta)' d\theta \quad (\text{部分積分を使う}) \\
 &= \left[\frac{\cos \theta}{\sin^3 \theta} \sin \theta \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{\cos \theta}{\sin^3 \theta} \right)' \sin \theta d\theta \\
 &= \left[\frac{\cos \theta}{\sin^2 \theta} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{-\sin^2 \theta - 3 \cos^2 \theta}{\sin^4 \theta} \sin \theta d\theta \\
 &= (0 - \sqrt{2}) + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin \theta} d\theta + 3 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 \theta}{\sin^3 \theta} d\theta \\
 &= -\sqrt{2} - \log (\sqrt{2} - 1) + 3 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 \theta}{\sin^3 \theta} d\theta
 \end{aligned}$$

以上より $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 \theta}{\sin^3 \theta} d\theta = \frac{\sqrt{2} + \log (\sqrt{2} - 1)}{2}$ です。