2019 年千葉大学問題 *12*

$\displaystyle\int_0^{\frac{\pi}{3}} \tan^7\theta\, d\theta$ を計算してください。

解説・解答

n は自然数とします。 $(\tan\theta)' = \dfrac{1}{\cos^2\theta} = 1 + \tan^2\theta$

$$
\begin{aligned}
\int_0^{\frac{\pi}{3}} \tan^{n+2}\theta\,d\theta &= \int_0^{\frac{\pi}{3}} (\tan^n\theta + \tan^{n+2}\theta)\,d\theta - \int_0^{\frac{\pi}{3}} \tan^n\theta\,d\theta \\
&= \int_0^{\frac{\pi}{3}} \tan^n\theta(1+\tan^2\theta)\,d\theta - \int_0^{\frac{\pi}{3}} \tan^n\theta\,d\theta \\
&= \int_0^{\frac{\pi}{3}} \tan^n\theta(\tan\theta)'\,d\theta - \int_0^{\frac{\pi}{3}} \tan^n\theta\,d\theta \\
&= \left[\frac{\tan^{n+1}\theta}{n+1}\right]_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} \tan^n\theta\,d\theta \\
&= \frac{3^{\frac{n+1}{2}}}{n+1} - \int_0^{\frac{\pi}{3}} \tan^n\theta\,d\theta
\end{aligned}
$$

$$\int_0^{\frac{\pi}{3}} \tan\theta\,d\theta = \int_0^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta}\,d\theta = \int_0^{\frac{\pi}{3}} \frac{-(\cos\theta)'}{\cos\theta}\,d\theta = \Big[-\log|\cos\theta|\Big]_0^{\frac{\pi}{3}} = -\log\frac{1}{2} = \log 2$$

$$\int_0^{\frac{\pi}{3}} \tan^3\theta\,d\theta = \frac{3}{2} - \int_0^{\frac{\pi}{3}} \tan\theta\,d\theta = \frac{3}{2} - \log 2$$

$$\int_0^{\frac{\pi}{3}} \tan^5\theta\,d\theta = \frac{3^2}{4} - \int_0^{\frac{\pi}{3}} \tan^3\theta\,d\theta = \frac{9}{4} - \left(\frac{3}{2} - \log 2\right) = \frac{3}{4} + \log 2$$

$$\int_0^{\frac{\pi}{3}} \tan^7\theta\,d\theta = \frac{3^3}{6} - \int_0^{\frac{\pi}{3}} \tan^5\theta\,d\theta = \frac{9}{2} - \left(\frac{3}{4} + \log 2\right) = \frac{15}{4} - \log 2$$